Chapter 2

First-Order Equations

This chapter concerns solving differential equations of the form

$$\frac{dy}{dt} = f(t, y), \tag{2.1}$$

where $f(t, y)$ and the partial derivative $f_y(t, y)$ are continuous. How you solve this equation depends on how $f(t, y)$ depends on y, and this is explained in Sections 2.1 and 2.2. In fact, you can often determine properties of the solution without actually solving the equation, and what this entails is introduced in Section 2.4.

2.1 ▪ Separable Equations

To introduce this method we begin by considering the differential equation

$$\frac{dy}{dt} = 3y^2. \tag{2.2}$$

We are going to treat the derivative as if it were a fraction, and rewrite the above equation as

$$\frac{dy}{y^2} = 3dt. \tag{2.3}$$

So, the variables have been separated in the sense that all of the y terms are on the left hand side, and the t terms are on the right. We now integrate both sides, which gives

$$\int \frac{dy}{y^2} = \int 3dt.$$

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Carrying out the integrations, and including the usual integration constant, we have

$$-\frac{1}{y} = 3t + c. \tag{2.4}$$

Solving this for y, we obtain the solution

$$y = -\frac{1}{3t+c}. \tag{2.5}$$

The last step is to check on whether the separation of variables step might involve dividing by zero. This happens for (2.3) when $y = 0$. Moreover, the constant function $y = 0$ is a solution of (2.2), and it is not included in (2.5). Consequently, another solution of the differential equation is

$$y = 0. \tag{2.6}$$

The method used to solve (2.2) is rather simple, but it contains the questionable step of separating the derivative to obtain (2.3). To explain why this is possible, note that (2.2) can be written as $y^{-2}\frac{dy}{dt} = 3$. Using the chain rule, this can be written as $-\frac{d}{dt}(y^{-1}) = 3$. Integrating this equation yields (2.4). So, separation of variables is a method based on solving the equation using the chain rule.

2.1.1 ▪ General Version

To explain how the method can be used for other problems, suppose the differential equation to solve is

$$\frac{dy}{dt} = f(t, y). \tag{2.7}$$

The method requires that it is possible to find a factorization of the form $f(t, y) = F(t)G(y)$. This means that it is possible to write the differential equation as

$$\frac{dy}{dt} = F(t)G(y). \tag{2.8}$$

Separating variables gives

$$\frac{dy}{G(y)} = F(t)dt,$$

and integrating we get

$$\int \frac{dy}{G(y)} = \int F(t)dt. \tag{2.9}$$

In theory, you carry out the above integrations, and then solve for y. How difficult this might be depends on how complicated the y integral is, and the examples that follow illustrate some of the complications that can arise. It is also important to note that the above method requires that $G(y) \neq 0$. Consequently, in addition to the solutions that come from (2.9), you must include as solutions any constant that satisfies $G(y) = 0$.

Example 1: Find all solutions, obtained using separation of variables, of $4y' = -y^3$.

Answer: Since $f(t, y) = -\frac{1}{4}y^3$, we can take $F(t) = \frac{1}{4}$ and $G(y) = -y^3$. So, (2.9) becomes

$$-\int \frac{dy}{y^3} = \int \frac{1}{4} dt.$$

Integrating gives us

$$\frac{1}{2y^2} = \frac{1}{4}t + c,$$

which is rewritten as

$$y^2 = \frac{2}{t + 4c}.$$

From this we obtain the two solutions

$$y = \pm\sqrt{\frac{2}{t + 4c}}. \tag{2.10}$$

To check on the $G(y) = 0$ solutions, solving $G(y) = 0$ gives $y = 0$. This constant function is not included in the above expressions for y, so it is a third solution of the equation. ■

Example 2: Find the solution of the IVP: $4y' = -y^3$, where $y(0) = -3$.

Answer: The three solutions of the differential equation were derived in the previous example. Because the initial condition requires the solution to be negative, the solution we need is

$$y = -\sqrt{\frac{2}{t + 4c}}.$$

Setting $y = -3$ and $t = 0$ in this equation gives $3 = 1/\sqrt{2c}$, which means that $c = 1/18$. Therefore, the solution is

$$y = -\sqrt{\frac{18}{9t + 2}}. \quad ■$$

Example 3: Is $y' + y = t$ a separable equation?

Answer: No. For this equation, $f(t, y) = t - y$, and it is not possible to factor this as $f(t, y) = F(t)G(y)$. How to solve this equation is explained in the next section. ■

Example 4: Solve $\frac{dw}{dx} = \frac{x}{1+w}$, where $w(0) = -2$.

Answer: In this problem the independent variable is x and the dependent variable is w. Separating variables, so $(1+w)dw = xdx$, and then integrating gives

$$\int (1+w)dw = \int xdx.$$

Carrying out the integrations we get that

$$w + \frac{1}{2}w^2 = \frac{1}{2}x^2 + c.$$

To satisfy the initial condition, substitute $w = -2$ and $x = 0$ into the above equation, from which we get that $c = 0$. This leaves $w + \frac{1}{2}w^2 = \frac{1}{2}x^2$, or equivalently, $w^2 + 2w - x^2 = 0$. This is a quadratic equation in w, and solving it we get the two solutions

$$w = -1 \pm \sqrt{1+x^2}.$$

The initial condition is needed to determine which sign to use, and since $w(0) = -2$ then we need the minus sign. Therefore, the solution of the IVP is $w = -1 - \sqrt{1+x^2}$. ■

Example 5: Solve $y' = -\frac{y}{1+y}$, where $y(0) = 1$.

Answer: Separating variables yields

$$\frac{1+y}{y}dy = -dt.$$

Since $(1+y)/y = 1/y + 1$, and $y(0) > 0$, then integrating we get that

$$y + \ln y = -t + c.$$

It is not possible to solve this for y as in the previous examples, without resorting to more advanced mathematical methods. For this reason, this is an example of what is called an *implicit solution*, and they are very common when solving nonlinear differential equations. Even so, it is still possible to find c from the initial condition. Substituting $y = 1$ and $t = 0$ into the above equation we get that $c = 1$. Therefore, the solution of the IVP is defined implicitly through the equation

$$y + \ln y = -t + 1. \quad ■ \tag{2.11}$$

A few comments need to be made about separation of variables before ending this section.

Integration Constant: The integration constant plays an essential role in the solution of a differential equation. It is useful to be aware that there are different ways you can write it. As an example, instead of (2.10), you can write the solution as

$$y = \pm\sqrt{\frac{2}{t + \bar{c}}}\,,$$

where $\bar{c} = 4c$. Similarly, if the solution is found to be

$$y = \frac{3t - 2c + 4}{t + 2c - 4}\,,$$

you can write it as

$$y = \frac{3t - \bar{c}}{t + \bar{c}}\,, \tag{2.12}$$

where $\bar{c} = 2c - 4$. For both of these examples, the solution contains one undetermined constant, just as in the original version of each solution. It should also be mentioned that this simplification is often used when giving the answers to the exercises. Moreover, instead of (2.12), the answer will likely be written as

$$y = \frac{3t - c}{t + c}\,.$$

Linear or Nonlinear: The method works on linear and nonlinear first-order differential equations. However, it does not work on every linear or nonlinear equation.

Non-uniqueness of a Factorization: The functions $F(t)$ and $G(y)$ in the factorization $f(t, y) = F(t)G(y)$ are not unique. For example, for $f(t, y) = y + ty$ you can take $F(t) = 1 + t$ and $G(y) = y$. You can also take $F(t) = \frac{1}{2}(1+t)$ and $G(y) = 2y$. It makes no difference which one you use, it is just required that $f(t, y) = F(t)G(y)$. Any such factorization will lead, eventually, to the same, or an equivalent, solution of the differential equation.

Existence and Uniqueness: When solving an IVP there is always the question of whether there is a solution (existence), or whether there is more than one solution (uniqueness). As it turns out, there are problems that have no solution (see Exercise 6(a)), or have multiple solutions (see Exercise 6(b)). Preventing such situations is the reason for assuming that $f(t, y)$ and $f_y(t, y)$ are continuous in (2.1). The formal statement of the requirements, and the conclusions, are contained in what is known as the Picard-Lindelöf theorem. However, the assumed continuity does not guarantee that there is a solution that holds for $0 \le t < \infty$ (see Exercise 6(c)).

Exercises

1. Find all of the solutions, obtained using separation of variables, of the given differential equation.

   a) $y' = -3y^4$
   b) $y' = y^3 e^{-t}$
   c) $y' + y^2 \sin t = 0$
   d) $2y' = t/(y-3)$
   e) $y' = -(2+t)e^y$
   f) $(1+t)y' = -e^{3y}$
   g) $y' = -e^{2t+4y}$
   h) $y' = -2^y$
   i) $y' + (1+3y)^3 = 0$
   j) $y' = y^2 + 4y + 4$
   k) $2y' = y^2 - 6y + 9$
   l) $3y' = y^2 + 1$
   m) $y' + te^y = te^{-y}$
   n) $y' - e^{-y} = 1$
   o) $y' = t(y + 1/y)$

2. Find the solution of the IVP.

   a) $y' = -3y^3, \quad y(0) = 5$
   b) $y' = -2y^3, \quad y(0) = 0$
   c) $(1+t)y' = 3 + y, \quad y(0) = 4$
   d) $(4 + e^t)y' + e^t y^2 = 0$, $y(0) = 1$
   e) $y' = te^{-y}, \quad y(0) = -1$
   f) $y' = \dfrac{1}{2+y}, \quad y(0) = 0$
   g) $y' = 1 + \cos(y), \quad y(0) = \pi/2$
   h) $y' = y^2 - 5y, \quad y(0) = 1$
   i) $y' + e^{-2y} = 1, \quad y(0) = 1$
   j) $y' = 1/(e^{-y} + e^y), \quad y(0) = 0$
   k) $y' = \sqrt{1 - y^2}, \quad y(0) = 0$
   Hint: $y' \geq 0$

3. Find the solution of the IVP. In these problems, the independent variable is not t and the dependent variable is not y.

   a) $\dfrac{dq}{dr} = -7q^3, \quad q(0) = -1$
   b) $\dfrac{dp}{dr} = -4p^3, \quad p(0) = 0$
   c) $3\dfrac{dh}{d\tau} = 2 + h, \quad h(0) = 2$
   d) $\dfrac{dh}{dx} = h^2 - 3h, \quad h(0) = 2$
   e) $(1+e^{-r})\dfrac{dz}{dr} + z^2 = 0, \quad z(0) = 6$
   f) $4\dfrac{dw}{d\tau} = \tau^3 e^{-2w}, \quad w(0) = 0$
   g) $(\theta + 1)^3 \dfrac{dr}{d\theta} = r^2, \quad r(0) = 2$
   h) $\dfrac{dr}{d\theta} = \dfrac{2\theta}{1+r}, \quad r(0) = 0$

4. Find the solution of the IVP in implicit form.

   a) $y' = 1 + \dfrac{1}{y}, \quad y(0) = 1$
   b) $y' = \dfrac{3}{1+y^4}, \quad y(0) = -1$
   c) $y' = \dfrac{1+y}{2+y}, \quad y(0) = 5$
   d) $\dfrac{dp}{dr} = \dfrac{e^p}{1+e^p}, \quad p(0) = 2$

5. A cable is hung between two poles as illustrated in Figure 2.1. The poles are located at $x = -L$ and $x = L$, and each has height h.

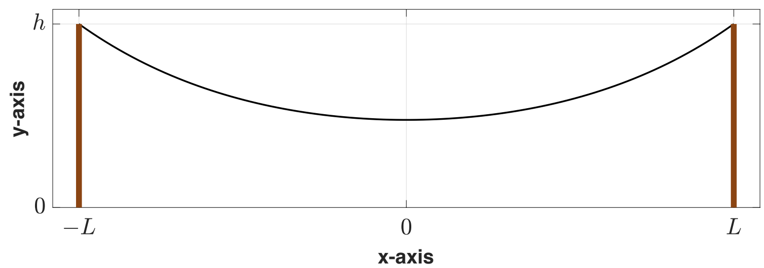

Figure 2.1. *Cable hanging between two poles, as described in Exercise 5.*

The curve $y(x)$ minimizes the cable's potential energy. From this, one obtains the equation

$$a\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}, \quad \text{for } -L < x < L,$$

where a is a positive constant. Because of the symmetry in the problem, $y'(0) = 0$.

a) Letting $w(x) = y'(x)$, rewrite the differential equation as a first-order equation involving w and w'. Also, what is $w(0)$?
b) Solve the problem in part (a) for w.
c) Integrate $y'(x) = w(x)$, and use the condition $y(L) = h$, to determine $y(x)$. The solution you are finding is an example of what is called a catenary.

6. The following illustrate some of the complications that can arise when solving differential equations.

a) Consider the IVP: $ty' = y + 1$, where $y(0) = 1$. Try solving this and show that there is no solution (at least when using separation of variables). Note that this equation does not satisfy the continuity assumptions made for (2.1).
b) Show that there are an infinite number of solutions of $ty' = y + 1$, where $y(0) = -1$.
c) Solve $y' = \frac{1}{2}y^3$, where $y(0) = 1$. Explain why there is no solution for $t \geq 1$. This is known as blowup in finite time.

2.2 ▪ Linear Equations

The equation to be solved is

$$y' + p(t)y = g(t). \tag{2.13}$$

What is important here is that this equation is linear, as well as first-order. Also, our earlier assumption that $f(t, y)$ and the partial derivative